How to create a problem for the long test?

Rules.

1. Create a problem.
2. Send it to me by e-mail, or just hand it to me. The deadline is our class on Tuesday (17.12.2019).
3. On Tuesday, I will publish all suggestions on my website.
4. On Thursday (19.12.2019), you will get one of the suggested examples as a problem in the long test.

Points 3 and 4 will happen **only if** I decide I have received enough reasonable propositions.

What kind of problems? Given a square matrix A, find its eigenvalues and the corresponding eigenspaces. Decide whether A is diagonalizable. If it is, find C such that $C^{-1} \cdot A \cdot C$ is diagonal, and compute this diagonal matrix.

Alternatively, for a given linear map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ (defined by a formula or by a matrix $[\varphi]_{\mathrm{st}}^{\mathrm{st}}$), find its eigenvalues and the corresponding eigenspaces. Decide whether φ is diagonalizable. If it is, find a basis $\mathcal{A}$ such that $[\varphi]_{\mathcal{A}}^{\mathcal{A}}$ is diagonal, and compute this diagonal matrix.

How to do it?

1. Choose a diagonal matrix D (or an almost-diagonal matrix, read below) and an invertible matrix C.
2. Find C^{-1}.
3. Compute $A = C \cdot D \cdot C^{-1}$.
4. That's it! Now the matrix A should be a reasonable example to work on. Alternatively, φ given by the matrix $[\varphi]_{\mathrm{st}}^{\mathrm{st}} = A$ (or by the formula associated with this matrix) is a good choice for a map.

How to do it with Wolfram Alpha? Once you pick D and C, you can use ask Wolfram Alpha to compute the rest. For example take $C = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$ and type

```
determinant of { {4,3}, {3,2} }
```

This way we check that C has determinant -1, hence it does have an inverse. Let's find it:

```
inverse of { {4,3}, {3,2} }
```

Now we know that $C^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$. You can get this answer in text format by placing your mouse cursor below *Results* and choosing option *Plain text*. Note that if $\det C$ is different from ± 1, you will get some fractions in C and probably also in A.

Let us choose $D = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. Now type

```
{ {4,3}, {3,2} } * { {-1,0}, {0,0} } * inverse[{ {4,3}, {3,2} }]
```

or

```
{ {4,3}, {3,2} } * { {-1,0}, {0,0} } * { {-2,3}, {3,-4} }
```

to get $A = \begin{bmatrix} 8 & -12 \\ 6 & -9 \end{bmatrix}$. Once again, you may copy this answer to the clipboard. To check everything is all right, type

```
diagonalize {{8,-12}, {6,-9}}
```

and you should see the eigenvalues we started with. The matrix S given by Wolfram is not necessarily the same as C, because possibly there are many choices of bases for the eigenspaces.

If you ask to diagonalize matrix that is not diagonalizable, Wolfram will tell you it is not possible, and suggest *Jordan decomposition* instead. Try for example:

```
diagonalize {{2,1}, {0,2}}
```

Here you can see what happens if you type it:
https://www.wolframalpha.com/input/?i=determinant+of+%7B+%7B4%2C3%7D%2C+%7B3%2C2%7D+%7D
https://www.wolframalpha.com/input/?i=inverse+of+%7B+%7B4%2C3%7D%2C+%7B3%2C2%7D+%7D
https://www.wolframalpha.com/input/?i=%7B+%7B4%2C3%7D%2C+%7B3%2C2%7D+%7D+*+%7B+%7B-1%2C0%7D%2C+%7B0%2C0%7D+%7D+*+inverse%5B%7B+%7B4%2C3%7D%2C+%7B3%2C2%7D+%7D%5D
https://www.wolframalpha.com/input/?i=%7B+%7B4%2C3%7D%2C+%7B3%2C2%7D+%7D+*+%7B+%7B-1%2C0%7D%2C+%7B0%2C0%7D+%7D+*+%7B+%7B-2%2C3%7D%2C+%7B3%2C-4%7D+%7D
https://www.wolframalpha.com/input/?i=diagonalize+%7B%7B8%2C-12%7D%2C+%7B6%2C-9%7D%7D+
https://www.wolframalpha.com/input/?i=diagonalize+%7B%7B2%2C1%7D%2C+%7B0%2C2%7D%7D+

How to make it interesting?

1. **Repeat some eigenvalue.**
2. **Use block matrices.** If you choose C as a block matrix, e.g.,

$$C = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

   the resulting A will also be a block matrix:

$$A = \begin{bmatrix} 8 & -12 & 0 & 0 \\ 6 & -9 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

   Although it is 4×4, it is still easily solvable.

3. **Choose a non-diagonal matrix D instead.** If you repeat some eigenvalue λ in D, and then add a "1" in a row and column in which this λ appears, for example

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{or} \quad D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

   you obtain a non-diagonal D. Moreover, such D is not diagonalizable (check yourself!). Whatever C you choose, the resulting matrix $A = C \cdot D \cdot C^{-1}$ will **not be diagonalizable**.

   Note that $D = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ does not work. Such D is not diagonal, but still diagonalizable.